Fixed point theorems for contractivity widely more generalized hybrid mappings in metric spaces (Nonlinear Analysis and Convex Analysis)

数理解析研究所講究録
第2011巻 2016年 172-177

Fixed point theorems for contractively widely more generalized hybrid mappings in metric spaces

日本大学　川﨑敏治 (toshiharu.kawasaki@nifty.ne.jp)
(Toshiharu Kawasaki, Nihon University)

Abstract

In this paper we consider a broad class of mappings containing Kannan mappings and contratively generalized hybrid mappings. Then we deal with fixed point theorems for such a mapping. Using these results, we show directly well-known fixed point theorems in complete metric spaces.

1 Introduction

Let (X, d) be a metric space. A mapping T from X into itself is said to be contractive if there exists k wirh $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for any $x, y \in X$. Such a mapping is called a k-contractive mapping. A mapping T from X into itself is said to be Kannan [5] if there exists k with $k \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty))$$

for any $x, y \in X$. A mapping T from X into itself is said to be contractively nonspreading [1,4,9] if there exists k with $k \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx))$$

for any $x, y \in X$. A mapping T from X into itself is said to be contractively hybrid [3] if there exists k with $k \in \left[0, \frac{1}{3}\right)$ such that

$$d(Tx, Ty) \leq k(d(Tx, y) + d(Ty, x) + d(x, y))$$

for any $x, y \in X$. Recently, Hasegawa, Komiya and Takahashi [3] introduced the concept of contratively generalized hybrid mappings on metric spaces and studied the fixed point theorems for such mappings on complete metric spaces. A mapping T from X into itself is said to be contratively generalized hybrid if there exist $\alpha, \beta, r \in \mathbb{R}$ with $r \in [0,1)$ such that

$$\alpha d(Tx, Ty) + (1-\alpha)d(x, Ty) \leq r(\beta d(Tx, y) + (1-\beta)d(x, y))$$

for any $x, y \in X$. Such a mapping is called an (α, β, r)-contratively generalized hybrid mapping; see also Kocourek, Takahashi and Yao [7] for such a mapping in Hilbert spaces. For instance, if $\alpha = 1$ and $\beta = 0$, then an (α, β, r)-contratively generalized hybrid mapping is contractive; if $\alpha = 1 + r$ and $\beta = 1$, then an (α, β, r)-contratively generalized hybrid mapping is contractively nonspreading; if $\alpha = 1+\frac{r}{2}$ and $\beta = \frac{1}{2}$, then an (α, β, r)-contratively generalized hybrid mapping is contractively hybrid; see Hasegawa, Komiya and Takahashi [3].

In this paper, motivated by Hasegawa, Komiya and Takahashi [3], we consider a broad class of mappings containing Kannan mappings and contratively generalized hybrid mappings. Then we deal with fixed point theorems for such a mapping. Using these results, we show directly well-known fixed point theorems in complete metric spaces.

2 Preliminaries

We know the following Caristi's fixed point theorem which was generalized by Takahashi [8].

Theorem 2.1. *Let (X, d) be a complete metric space, let ψ be a proper, bounded below, and lower semicontinuous mapping from X into $(-\infty, \infty]$, and let T be a mapping from X into itself. Suppose that*

$$d(x, Tx) + \psi(Tx) \leq \psi(x)$$

for any $x \in X$. Then T has a fixed point.

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$, which is the dual space of ℓ^∞. Then we denote by $\mu(x)$ the value of μ at $x = (x_1, x_2, \dots) \in \ell^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(x)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, \dots)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n$$

holds for any $x = (x_1, x_2, \dots) \in \ell^\infty$. In particular, if $x = (x_1, x_2, \dots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we obtain $\mu_n(x_n) = a$. See [8] for the proof of existence of a Banach limit and its other elementary properties.

Moreover we use the following lemma and theorem showed by Hasegawa, Komiya and Takahashi [3].

Lemma 2.1. *Let (X, d) be a metric space, let $\{x_n\}$ be a bounded sequence in X, let μ be a mean on ℓ^∞ and let g be a mapping from X into $\mathbb{R}$ defined by*

$$g(x) = \mu_n d(x_n, x)$$

for any $x \in X$. Then g is continuous.

Theorem 2.2. *Let (X,d) be a complete metric space, let μ be a mean on ℓ^∞ and let T be a mapping from X into itself. Suppose that there exist a real number r with $0 \leq r < 1$ and $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and*

$$\mu_n d(T^n z, Tx) \leq r \mu_n d(T^n z, x)$$

for any $x \in X$. Then the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

3 Fixed point theorems

In this section we consider an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-contractively widely more generalized hybrid mapping from a metric space X into itself; see also Kawasaki and Takahashi [6] for such a mapping in Hilbert spaces.

Definition 3.1. Let (X,d) be a metric space and let T be a mapping from X into itself. We say that T is contractively widely more generalized hybrid if T satisfies the following condition: there exist real numbers α, β, γ, δ, ε and ζ such that

$$\alpha d(Tx,Ty) + \beta d(x,Ty) + \gamma d(Tx,y) + \delta d(x,y) + \varepsilon d(x,Tx) + \zeta d(y,Ty) \leq 0$$

for any $x, y \in X$. Such a mapping T is called an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-contractively widely more generalized hybrid mapping.

Firstly we consider criteria for an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-contractively widely more generalized hybrid mapping T from a metric space X into itself such that $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

Lemma 3.1. *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B1), (B2) *or* (B3)*:*

(B1) *$\alpha + \beta + \zeta \geq 0$ and $\alpha + 2\min\{\beta, 0\} + \delta + \varepsilon + \zeta > 0$;*

(B2) *$\alpha + \gamma + \varepsilon \geq 0$ and $\alpha + 2\min\{\gamma, 0\} + \delta + \varepsilon + \zeta > 0$;*

(B3) *$2\alpha + \beta + \gamma + \varepsilon + \zeta \geq 0$ and $\alpha + \min\{\beta + \gamma, 0\} + \delta + \varepsilon + \zeta > 0$.*

Then $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

Using Lemma 3.1, we obtain directly the following theorem.

Theorem 3.1. *Let (X,d) be a complete metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B1), (B2) *or* (B3)*. Then for any $x \in X$ there exists $\lim_{n\to\infty} T^n x$.*

Remark 3.1. Let (X, d) be a metric space and let $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ be a Cauchy sequence in X. Then $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Indeed, since $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence, for any positive number ρ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \rho$ for any $m, n \geq N$. Put $M = \max\{d(x_0, x_N), \dots, d(x_{N-1}, x_N), \rho\}$. Then $d(x_n, x_N) \leq M$ for any $n \in \mathbb{N} \cup \{0\}$.

Using Theorem 2.1, we show the following fixed point theorem.

Theorem 3.2. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (C1), (C2) *or* (C3):

(C1) $\zeta > 0$, $\alpha + \beta \geq 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} \geq 0$;

(C2) $\varepsilon > 0$, $\alpha + \gamma \geq 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\zeta, 0\} \geq 0$;

(C3) $\varepsilon + \zeta > 0$, $2\alpha + \beta + \gamma \geq 0$ *and* $\alpha + \beta + \gamma + \delta \geq 0$.

Then T has a fixed point if and only if there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Using Lemma 3.1, Remark 3.1 and Theorem 3.2, we obtain the following fixed point theorem.

Theorem 3.3. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying the following:*

(B) *one of* (B1), (B2) *and* (B3) *holds;*

(C) *one of* (C1), (C2) *and* (C3) *holds.*

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Using Theorem 2.2, we show the following fixed point theorem.

Theorem 3.4. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (H1), (H2) *or* (H3):

(H1) $\alpha + \beta + \zeta > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$;

(H2) $\alpha + \gamma + \varepsilon > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$;

(H3) $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon + \zeta, 0\} > 0$.

Then T has a fixed point if and only if there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Moreover the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Using Lemma 3.1, Remark 3.1 and Theorem 3.4, we obtain the following fixed point theorem.

Theorem 3.5. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying the following:*

(B) *one of* (B1), (B2) *and* (B3) *holds;*

(H) *one of* (H1), (H2) *and* (H3) *holds.*

Then the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Moreover, if (B) is satisfied, we also show the following fixed point theorem.

Theorem 3.6. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B), *and one of* (M1), (M2) *and* (M3)*:*

(M1) $\alpha + \beta + \zeta > 0$;

(M2) $\alpha + \gamma + \varepsilon > 0$;

(M3) $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$.

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

4 Applications

Theorem 4.1. *Let (X, d) be a complete metric space and let T be a contractively generalized hybrid mapping form X into itself, that is, there exist $\alpha, \beta, r \in \mathbb{R}$ with $0 \le r < 1$ such that*

$$\alpha d(Tx, Ty) + (1-\alpha) d(x, Ty) \le r(\beta d(Tx, y) + (1-\beta) d(x, y))$$

for any $x, y \in X$. Suppose that $\alpha > r(1 + |\beta|)$. Then the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) $\quad u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

Theorem 4.2. *Let* (X, d) *be a complete metric space and let* T *be a mapping form* X *into itself satisfying there exist* $\varepsilon, \zeta \in \mathbb{R}$ *such that* $\varepsilon + \zeta < 1$ *and*

$$d(Tx, Ty) \leq \varepsilon d(x, Tx) + \zeta d(y, Ty)$$

for any $x, y \in X$. *Then the following hold:*

(i) $\quad T$ *has a unique fixed point* $u \in X$;

(ii) $\quad u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

References

[1] S. K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[2] K. Hasegawa, T. Kawasaki, and W. Takahashi, *Fixed point theorems for contractively widely more generalized hybrid mappings in metric spaces*, Linear and Nonlinear Analysis, to appear.

[3] K. Hasegawa, T. Komiya, and W. Takahashi, *Fixed point theorems for generalized contractive mappings in metric spaces and estimating expressions*, Scientiae Mathematicae Japonicae **74** (2011), 15–27.

[4] S. Iemoto, W. Takahashi, and H. Yingtaweesittikul, *Nonlinear operators, fixed points and completeness of metric spaces*, Fixed Point Theory and its Applications (L. J. Lin, A. Petrusel, and H. K. Xu, eds.), Yokohama Publishers, Yokohama, 2010, pp. 93–101.

[5] R. Kannan, *Some results on fixed points*, Amer. Math. Monthly **76** (1969), 405–408.

[6] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 71–87.

[7] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese Journal of Mathematics **14** (2010), 2497–2511.

[8] W. Takahashi, *Nonlinear Functional Analysis. Fixed Points Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

[9] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. (Basel) **23** (1972), 292–298.